Macroeconomic Prelim Prep

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1 A Solow Model Question

Consider a standard neoclassical (Solow) growth model. The production function is given by:

$$Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$$

The model defines population grows at rate n, and technology grows at rate g. A constant fraction s of output is invested each period, and a constant fraction δ of the capital stock depreciates each period.
In what follows, "steady state" refers to the steady state in terms of effective units of labor [1].

a) *There are two main implications of the Solow model. What are they?*

   (a) The Solow model has decreasing returns to capital. This implies countries with very little capital to grow faster than richer ones. As a result the growth rate of capital per worker decreases as the level of capital per worker increases. (Because it is converging to a constant level at steady state. This is defined as convergence in the growth literature.

   (b) The growth rate of output per capita in the steady state only depends on g, which is exogenous to the model.

b) *Write down the equation for the evolution of the capital stock per effective unit of labor.*

$$\dot{k} = sf(k) - (n + g + \delta)k$$

The change in capital stock is equal the fraction of output saved minus the fraction of capital per effective labor unit due to population growth, technological progress and depreciation. Basically this is saying how much capital is being saved subtracted what being used.

c) *What is the steady state level of output ?*

Using the fact that in steady state , $\dot{k} = 0$,

$$sf(k) = (n + g + \delta)k$$

Plug in $f(k) = k^{\alpha}$ and then solve for K.

$$sk^{\alpha} = (n + g + \delta)k$$

$$k = \frac{s}{n + g + \delta}^{\frac{1}{1-\alpha}}$$

Then we get y by plugging in the steady state value of capital per ELU.

$$y = k^{\alpha} = \frac{s}{n + g + \delta}^{\frac{\alpha}{1-\alpha}}$$

[1] This just means divided by AL

d) *In the steady state, what is the growth rate of per capita output?*
There are a few key components to answering this problem. In the Solow model $k = K/AL$ is constant in the steady state. In other words the growth rate of k, (denoted as g_k is equal to zero). Intuitively this implies K must grow at the same rate as A and L. By definition of the model $g_A = g$ and $g_L = n$, so the growth rate of capital has to be $g + n$. Mathematically we find the growth rate by taking logs and the first derivative.

$$\ln k = \frac{\ln K}{lnAL} = \ln K - \ln A - \ln L$$

Take the derivative to get the growth rate

$$\frac{\dot{k}}{k} = g_k = \frac{\dot{K}}{K} - \frac{\dot{A}}{A} + \frac{\dot{L}}{L} = g_K - n + g$$

But we know because k is constant, $g_k = 0$ and

$$g_K = n + g$$

To answer the original question, you follow the same process for Y/L

$$\ln Y/L = \ln Y - \ln L$$

Take the derivative wrt to time to get the growth rate [2] :

$$g_{Y/L} = g_Y - n = g + n - n = g$$

We know the growth rate of output is the same as capital so we get that output per capita is equal to g, which is exogenous to the model.

e) *What is the Solow residual?*
The Solow Residual is all parts of output not explained by labor and capital (hence the word residual). The Solow Residual is often thought to reflect technological progress. Solow referred to it as a measure of our ignorance. Empirically estimates show it explains roughly 7/8 of growth in Y/L. Is is used in growth accounting literature to examine whether or not countries converge or not.

$$\frac{\dot{A}}{A} = \frac{\dot{Y}}{Y} - \alpha\frac{\dot{K}}{K} - (1-\alpha)\frac{\dot{L}}{L}$$

[2] $g_y = \frac{\dot{Y}}{Y} = \alpha g_K + (1-\alpha)(n+g) = g + n$

2 Continuous time investment with adjustment costs (2014 Final # 2)

Consider a representative firm that purchases capital at a constant price $p^k = 1$. This firm faces a capital adjustment costs of $C(I(t)) = \Phi(\frac{I(t)}{\kappa(t)})\kappa(t)$ where $\Phi^{'}(\cdot) > 0$, $\Phi^{''}(\cdot) \geq 0$, $\Phi(0) = 0$. In words, a firm must pay an increasing and convex cost of investment and capital does not depreciate so that $I(t) = \kappa(t)$. The real interest rate is constant and given by r. The representative firm's profits at time t (neglecting the cost of acquiring and adjusting capital) are proportional to its capital stock $\kappa(t)$ and decreasing in the industry-wide capital stock $K(t)$; hence they are given by $\pi(K(t))\kappa(t)$ where $\pi(K(t)) = a - bK(t)$ The firm maximizes the present value of profits

$$\Pi = \int_{t=0}^{\infty} e^{-rt}[\pi(K(t))\kappa(t) - p^K I(t) - C(I(t))]$$

a) *Set up the current value Hamiltonian and derive the first order conditions* [3].
The current value Hamiltonian is

$$H(t) = e^{-rt}[\pi(K(t))\kappa(t) - p^K I(t) - C(I(t))] + \lambda(t)I(t)$$

For the first order conditions, we maximize profits by choosing level of investment and capital, $I(t)$ and $\kappa(t)$, plus the shadow price.
For investment:

$$\frac{\partial H}{\partial I(t)} = -e^{-rt}[1 - C^{'}((I(t))] + \lambda(t) = 0$$

Now let's define $q(t) = \lambda e^{-rt}$ and solve for $q(t)$.

$$q(t) = 1 + C^{'}(I(t))$$

For the state variable, capital

$$-\frac{\partial H}{\partial \kappa(t)} = \dot{\lambda}(t) = -e^{-rt}[\pi(K(t))]$$

But note our definition for $q(t)$, $\lambda = e^{-rt}q(t)$, we can substitute and derive wrt to time.

$$e^{-rt}[\pi(K(t))] = \frac{\partial}{\partial t}[e^{-rt}q(t)] = -re^{-rt}q(t) + \dot{q}(t)e^{-rt}$$

Then simplify

$$\pi(K(t)) = rq(t) - \dot{q}(t)$$

For the shadow price, we get the budget constraint binds

$$\frac{\partial H}{\partial \lambda(t)} = \dot{\kappa}(t) = I(t)$$

b) *Interpret the first order conditions.*
$\frac{\partial H}{\partial I(t)}$ tells us that marginal cost of investment must be equal to the marginal benefit. $-\frac{\partial H}{\partial \kappa(t)} = \dot{\lambda}(t)$ states that in the dynamics the marginal cost of capital is equal to its marginal benefit and the third condition is the budget constraint.

c) *What is the equation for the $\dot{q} = 0$ locus? What is the long-run value of K (i.e., the value of K such that $\dot{q} = 0$ and $\dot{K} = 0$).*
So from our first order conditions we have the following,

$$\dot{q}(t) = rq(t) - \pi(K(t)) = 0$$

[3] This could also be solved using a value function, which would yield the same result. If you all want me to add that, let me know.

Then solve for $q(t)$. The fact that long run $I(t) = 0$ implies $q(t) = 1$

$$q(t) = \frac{\pi(K(t))}{r} = 1$$

d) *Use a phase diagram to illustrate the saddle path equilibrium long-run values of K and q.*

From implicit function theorem we know that $\dot{q} = 0$ is downward slopping, and in the last question I showed that where $\dot{K} = 0$ is constant. You would want to explain in which regions this does not converge (the bottom left and top right).

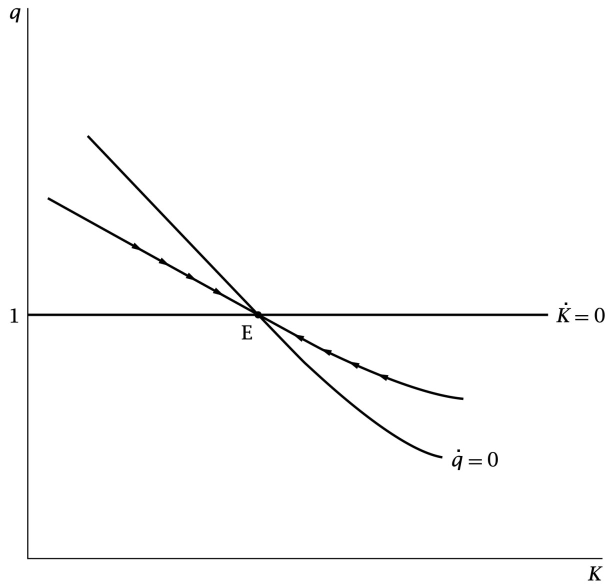


Figure 1: From Romer 2011 Advanced Macroeconomics pg 418

e) *Suppose that the price of acquiring capital increases from $p^k = 1$ to $p^k = 2$ Assume this change is permanent. Illustrate the new long-run equilibrium and the adjustment to this new equilibrium using a phase diagram. Explain the adjustment process in words*

Changing the price of acquiring changes the price in the original problem. This means the Hamiltonian is now

$$H(t) = e^{-rt}[\pi(K(t))\kappa(t) - 2I(t) - C(I(t))] + \lambda(t)I(t)$$

altering the FOC of investment such that $q(t) = 2 + C^{'}(I(t))$. Using the same logic as the part C, we get the $q(t) = 2$, and the phase diagram shifts the line of $\dot{K} = 0$ up, following the same path at a near relative level.

f) *Explain one reason why it is difficult to obtain an empirical measure of marginal q*

Answer:

So here is a passage directly from Romer (2011) pg. 426 : "Measuring marginal q (which is what the theory implies is relevant for investment) is extremely difficult; it requires estimating both the market value and the replacement cost of capital, accounting for a variety of subtle features of the tax code, and adjusting for a range of factors that could cause average and marginal q to differ."

My understanding of this in other terms is that there's a simultaneity and measurement error problem, meaning empirical measures from OLS will bias towards zero. Feel free to email me other things to add to this one.

3 Endogenous Growth (Prelim June 1018, # 3)

Consider a two-sector economy, where output is produced according to:

$$Y = [(1-a_K)K]^{\alpha}[A(1-a_L)L]^{1-\alpha},\ 0 < \alpha < 1$$

and ideas, or technology, are produced following:

$$\dot{A} = B(a_K K)^{\theta}(a_L L)^{\sigma} A^{\eta},\ B > 0\ ,\theta \geq 0,\ \sigma \leq 0$$

Also assume an exogenous savings rate and zero depreciation, so that $\dot{K} = sY$, and an exogenous growth rate of population: $\dot{L} = nL$.

a) *Explain why there is no a_A dividing technology between the two sectors.*
Ideas are nonrival. This means that the use of ideas in one sector does not prevent the use of ideas in another sector. Therefore there is no need to set total to proportional a_A and $(1-a_A)$ between the two sectors. Labor and capital are rival goods. The use of them in one sector prevents another sector from using it (Think you can't be two places at once).

b) *Find an expression for the growth rate of technology (stock of ideas) in terms of A, L, and K.*
This is one straightforward. We want the growth rate of technology, g_A. This is defined as $g_A = \frac{\dot{A}}{A}$. So we take the expression for $\dot{A}$ and divide by A to get

$$g_A = \frac{\dot{A}}{A} = B(a_K K)^{\theta}(a_L L)^{\sigma} A^{\eta-1}$$

c) *Find an expression for the growth rate of the capital stock in terms of A, L, and K.*
For this question we follow a same strategy. Note $g_K = \frac{\dot{K}}{K}$. So we are about to take the expression for $\dot{K}$ and divide by K.

$$g_K = \frac{\dot{K}}{K} = \frac{sY}{K}$$

$$g_K = s[(1-a_K)]^{\alpha}[(1-a_L)]^{1-\alpha}\left[\frac{AL}{K}\right]^{1-\alpha}$$

d) *Showing your work, solve for the balanced growth path (BGP) rates of capital and technology (g_k^* and g_A^* respectively).*
In order to find the balanced growth path rates we need to find when both technology and capital growth rates are constant. In other words we want g_k^* and g_A^* such that $\frac{\dot{g_A}}{g_A}$ and $\frac{\dot{g_K}}{g_K}$ are equal to zero.

Here I start with capital. To get $\frac{\dot{g_K}}{g_K}$, I take logs of the growth rate

$$\ln g_K = \ln s + \alpha \ln(1-a_K) + (1-\alpha)\ln(1-a_L) + (1-\alpha)(\ln A + \ln L - \ln K)$$

Then take the derivative with respect to time to get

$$\frac{\dot{g_K}}{g_K} = (1-\alpha)(\frac{\dot{A}}{A} + \frac{\dot{L}}{L} - \frac{\dot{K}}{K})$$

Then plug in the respective growth rates to get

$$\frac{\dot{g_K}}{g_K} = (1-\alpha)(g_A + n - g_k)$$

Then you repeat this process for technology/ideas to find $\frac{\dot{g_A}}{g_A}$

$$\ln g_A = \ln B + \theta(\ln a_k + \ln K) + \sigma(\ln a_L + \ln L - (1-\eta)\ln A$$

Then take the derivative with respect to time

$$\frac{\dot{g_A}}{g_A} = \theta g_K + \sigma n - (1-\eta) g_A$$

Now that we have expressions for the growth rates of the growth rates for technology and capital, we set them equal to zero to find the balanced growth path.

$$\theta g_K + \sigma n - (1-\eta) g_A = 0 \tag{3.1}$$

$$(1-\alpha)(g_A + n - g_k) = 0 \tag{3.2}$$

Since we have two equations and two unknowns, we can then easily solve for g_k^* and g_A^*. Start by simplifying equation 3.2 to the following:

$$g_k = g_A + n \tag{3.3}$$

Then plug into equation 3.1 and solve.

$$\theta(g_A + n) + \sigma n - (1-\eta) g_A = 0$$

Solve for g_A

$$g_A^* = \frac{(\theta+\sigma)n}{1-\eta-\theta}$$

and

$$g_K^* = \frac{(\theta+\sigma)n}{1-\eta-\theta} + n$$

e) *Given your answer to 3d, how fast does output per capita grow on the balanced growth path? Compare this to the Solow model.*
This question is asking for $g_{Y/L}$. To get this we take logs of output per capita and then derive with respect to time.

$$\ln Y/L = \alpha \ln(1-a_K) + (1-\alpha)(\ln(1-a_L) + \ln A) + \alpha(\ln K - \ln L)$$

$$g_{Y/L} = (1-\alpha) g_A + \alpha(g_K - n)$$

Then we plug in the result from equation 3.3.

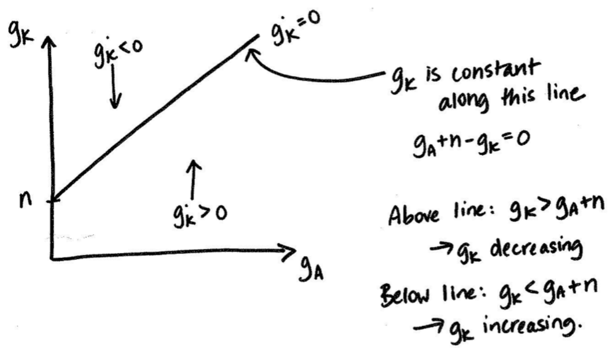
$$g_{Y/L} = (1-\alpha) g_A + \alpha(g_A + n - n)$$

$$g_{Y/L} = g_A$$

.

To answer the latter part of the question, I'll quote directly from Dr Minier's notes:
"This sounds a lot like the Solow model: long-run output per worker grows at the rate of technological progress. However, the difference is that we have explained that rate of technological progress as a function of the model's parameters in this case; in the Solow model, it was taken as an exogenous factor."

So now we need to think about what parameters endogenously determine g_A. Empirically this means we are able to compute comparative statistics for the relative parameters and see how they match reduced form estimates. Note this is a vary simple model, but it shows how you do not need a complex model to capture a lot of important points.
Below is a phase diagram of this relationship from Dr Minier's notes:



f) *To endogenously determine the equilibrium a_L, what would you do?*
We would need to set the marginal product of labor for the research sector and the output sector equal.

g) *Is the equilibrium a_L optimal? Why or why not?*
There is a number of reasons why the equilibrium a_L optimal may not be optimal. Individuals researchers do not take externalities into account. They do not account for their impact on L_A or A. In other words researchers do not know if they are increasing the labor research force productivity or hurting it (think you worked on a paper for 6 months, then someone else publishes the exact same thing in AER).

4 A Sample OLG Model

Consider an OLG model. Utility is CRRA (constant relative risk aversion) utility in each period, so that:

$$U_t = \frac{c_{1,t}^{1-\sigma}}{1-\sigma} + \beta \frac{c_{2,t+1}^{1-\sigma}}{1-\sigma}$$

The production function is Cobb-Douglas, where $F(K_t, L_t) = A_t K_t^{\alpha} L_t^{1-\alpha}$, and there is complete depreciation between periods($\delta = 1$). There is no technological progress (A is constant and set equal to one) and no population growth. As usual, agents have one unit of labor when young that they supply inelastically, earning wage w_t ; they do not work when old. Young agents can purchase $k_{t+1} > 0$ units of the next periods capital good, which provides them with a return of $k_{t+1} r_{t+1}$ when they are old (and then depreciates completely).

In the initial period (period 0) there are old individuals who only live in period 0. These old individuals are endowed with some amount Q $>$ 0 of the good.

a) *Write down the Lagrangian for the consumers problem and take first-order conditions.*

In order to do this we need to know what the two budget constraints are. First we have what you earn today is equal to how much you buy and how much you save [4] $w_t = c_{1,t} + k_{t+1}$. We also have the budget constraint for being old and not working, which is equal to your savings times the interest rate. $c_{2,t+1} = r_{t+1} k_{t+1}$.

$$w_t = c_{1,t} + k_{t+1} \tag{4.1}$$

$$c_{2,t+1} = r_{t+1} k_{t+1} \tag{4.2}$$

We can combine this into one constraint by plugging in the second into the first, and we write the Lagrangian as the following

$$\mathcal{L} = \frac{c_{1,t}^{1-\sigma}}{1-\sigma} + \beta \frac{c_{2,t+1}^{1-\sigma}}{1-\sigma} - \lambda_t \Big[c_{1,t} + \frac{c_{2,t+1}}{r_{t+1}} - w_t \Big]$$

Then we take derivatives with respect to the choice variables [5]

$$\frac{\partial \mathcal{L}}{\partial c_{1,t}} : c_{1,t}^{-\sigma} = \lambda_t$$

$$\frac{\partial \mathcal{L}}{\partial c_{2,t+1}} : \beta c_{2,t+1}^{-\sigma} = \frac{\lambda_t}{r_{t+1}}$$

b) *Find the Euler equation. When does the Euler equation hold?*

To do this we combine the results from the FOC to get the following:

$$\beta c_{2,t+1}^{-\sigma} = \frac{c_{1,t}^{-\sigma}}{r_{t+1}}$$

which we can simplify to

$$\frac{c_{2,t+1}}{c_{1,t}} = \beta r_{t+1}^{1/\sigma} \tag{4.3}$$

This dynamic equation always holds to relate the consumption of period t to period $t+1$ if markets clear and there are no externalities.

[4] for simplicity I will write these in equalities

[5] I leave out the budget constraint derivative

c) *Find expressions for the wage w_t and return to capital r_t. Firms are competitive and maximize profits.*

Because markets are perfectly competitive, factor prices are equal to their marginal product. In other words prices are equal to their first derivative to the production function.

$$\frac{\partial F(K_t, L_t)}{\partial K_t} = r_t = A_t K_t^{\alpha-1} L_t^{1-\alpha}$$
$$\frac{\partial F(K_t, L_t)}{\partial L_t} = A_t K_t^{\alpha} L_t^{-\alpha}$$

d) *Showing your work, find the savings rate in terms of model parameters.*

As a general role savings rate is saving/income. For an agent in this model that is

$$s_t = \frac{k_{t+1}}{w_t}$$

We can sub in the budget constraints from equations 4.1 and 4.2 to rewrite this in times of consumption.

$$s_t = \frac{c_{2,t+1}/r_{t+1}}{c_{1,t} + c_{2,t+1}/r_{t+1}}$$

Then we divide through by $c_{1,t}$ to get it in a form where we can use the Euler to get the savings rate in terms of model parameters (interest rate and discount factor)

$$s_t = \frac{c_{2,t+1}/c_{1,t}r_{t+1}}{1 + c_{2,t+1}/c_{1,t}r_{t+1}}$$

Then use the Euler (equation 4.3) to simplify

$$s_t = \frac{\beta^{1/\sigma} r_{t+1}^{\frac{1-\sigma}{\sigma}}}{1 + \beta^{1/\sigma} r_{t+1}^{\frac{1-\sigma}{\sigma}}}$$

e) *How would the savings rate you found in part 2d change in the case of log utility?*

Log is a special case of CRRA utility where $\sigma = 1$. So let $\sigma = 1$,

$$s_t = \frac{\beta}{1+\beta}$$

f) *How would the savings rate depend on the wage and interest rate if the utility function were linear?*

If utility was linear, you would only consume in the period in which consumption is worth more, so you either consume 0 or 1.

g) *How would the savings rate depend on the wage and interest rate if the utility function were Liontief?*

If utility was Liontief, you want to set consumption in both periods equal to each other. A simple plug in to part D would show you that

$$s_t = \frac{r_{t+1}}{1 + r_{t+1}}$$

5 Cash in Advance Model

Consider an endowment economy with homogeneous agents. The representative agent chooses a sequence for consumption, c_s, to maximize his lifetime discounted utility

$$\sum_{s=t}^{\infty} \beta^{s-t} u(c_s)$$

Each period the agent receives a real endowment of es at the beginning of period s and there is a linear technology that converts the endowment into the consumption good, so that $e_s = c_s$. The only financial asset in this economy is money. The monetary authority prints money and supplies it to the agents in the form of nominal lump-sum transfers, Ts; at the beginning of period s. Consumption must be financed out of cash on hand and transfers at the beginning of the period. In particular, the agent cannot use the $P_t e_t$ dollars he would receive for the sale of his endowment at s to finance consumption c_s: Let M_s be the nominal money holdings at the beginning of period s and P_s be the price level in period s. There are no taxes or government spending, all the government does is print money and transfer it to the households, thus the government budget constrain is $T_s = M_s - M_{s-1}$. Money grows at a constant rate, μ, so that $M_t = (1+\mu)M_{t-1}$

a) *Write the maximization problem of the representative agent.*

b) *Derive and interpret the first order conditions.*

c) *Consider the steady-state where the rate of money growth is given by μ; and the real variables are constant. Denote the steady-state value of a variable x_t as $\bar{x}$: Write the F.O.C. in the steady-state. Is money superneutral in the steady-state? Justify your answer.*

d) *Suppose you wanted to modify the model to consider two consumption goods, a "cash good", c_1; and a "credit good", c_2. Only the cash good is subject to a cash-in-advance constraint. The utility function in period s is given by $u = u(c_{1s}; c_{2s})$ where we assume $u_{11}; u_{22} < 0; \lim_{c_1 \to 0} u(c_1, c_2) = 1; \lim_{c_2 \to 0} u(c_1, c_2) = 1$, There is a linear technology for converting the endowment into either consumption good so that, $e_s = c_{1s} + c_{2s}$, which implies that in equilibrium the two consumption goods have the same price P_s: Write the maximization problem for the representative agent.*

e) *Derive the first order conditions and compare them to the conditions derived in (5b). Is money super-neutral in the steady-state in this model? Justify your answer.*

f) *Consider now the dynamic equilibrium, what would happen to the consumption of credit and cash goods if there is a permanent increase in the rate of growth of money in t + 1? Justify your answer.*

g) *An interpretation of cash and credit goods may be derived from a model where we assume that the individual consumes both leisure and consumption goods.Leisure doesn't require cash but consumption goods do. Thus suppose that the utility function is given by $u = u(c_s 1 - h_s)$ where h_s is hours worked and we have normalized total time to 1 so we can interpret $1 - h_s$ as leisure. There is no endowment, instead the individual earns a nominal wage W_t per hour worked and real profits in the form of dividends, d_t. Write the maximization problem of the representative agent. Given your answer to (5f), what would you expect to happen to the path of consumption and leisure if inflation increases in $t+1$?*

6 Durable Consumption June 2016 #4

Consider a representative consumer in a small open economy who has rational expectations about the future. The agent lives forever and derives utility from durable and nondurable consumption goods so that her expected lifetime utility at time t is given by

$$\sum_{s=t}^{\infty} \beta^{s-t}[\gamma \log C_s + (1-\gamma) \log D_s]$$

where C_s is the individual's consumption of nondurables on date s and D_s is the stock of durable goods the individual owns as period s ends. Durable goods depreciate at a rate δ; so that the law of motion for the stock of durable goods is given by

$$D_s = (1-\delta)D_{s-1} + C_s^D$$

where C_s^D denotes the individual's purchases of durable goods. The individual can trade foreign bonds, B_s with the rest of the world at an interest rate of r_s. In this pure endowment economy the agent receives a stochastic endowment Y_s every period and there is no government.

a) *Write the maximization problem for the representative consumer and interpret the period-to-period finance constraint. Make sure to define any variables you might need to introduce in order to set up the optimization problem.*

Since this is a pure endowment economy with no government, we do not have to worry about production or taxes. The agent is faced with the following period-to–period finance constraint

$$B_{s+1} + C_s + p_s C_s^D = (1+r_s)B_s + Y_s$$

where the left hand side of the equation is what is spent and the right is the "income". p_s is the price of a durable good. The consumer maximizes the original utility function subject to this constraint.

b) *Write the Bellman equation*

There are two state variables B_{t+1} and D_t. In order to write the Bellman we want to maximize the value function with respect to the state variables.

$$V_t(B_{t+1}, D_t) = \max_{B_{t+1}, D_t} \Big\{[\gamma \log C_t + (1-\gamma)\log D_t] + \beta V_{t+1}(B_{t+1}, D_t)\Big\} \tag{6.1}$$

Then we would want to plug in our budget constraint from part A, where we substitute in the durable good law of motion for C_s^D

$$B_{s+1} + C_s + p_s(D_s - (1-\delta)D_{s-1}) = (1+r_s)B_s + Y_s$$

Then plug in for the control variable C_s to get the answer

$$V_t(B_{t+1}, D_t) = \max_{B_{t+1}, D_t} \Big\{[\gamma \log \Big[(1+r_t)B_t + Y_t - B_{t+1} - (p_t(D_t - (1-\delta)D_{t-1}))\Big] + (1-\gamma)\log D_t] \\ + \beta V_{t+1}(B_{t+1}, D_t)\Big\}$$

c) *Derive the first order conditions.*

The first order conditions and back-substitution equation are [6]

$$\frac{\partial V_t}{\partial B_{t+1}} = \frac{-\gamma}{C_t} + \beta \frac{\partial V_{t+1}}{\partial B_{t+1}}$$

[6] I go ahead and plug back in for C_t

$$\frac{\partial V_t}{\partial D_t} = \frac{-\gamma p_t}{C_t} + \frac{1-\gamma}{D_t} + \beta\frac{\partial V_{t+1}}{\partial D_t}$$
$$\frac{\partial V_t}{\partial B_t} = \frac{\gamma(1+r_t)}{C_t}$$
$$\frac{\partial V_t}{\partial D_{t-1}} = \frac{\gamma p_t(1-\delta)}{C_t}$$

Then plug in the back substitution equations one period ahead to get

$$\frac{1}{C_t} = \beta\frac{(1+r_{t+1})}{C_{t+1}} \tag{6.2}$$
$$\frac{p_t}{C_t} = \beta\frac{p_{t+1}(1-\delta)}{C_{t+1}} + \frac{1-\gamma}{\gamma D_t} \tag{6.3}$$

Then solve equation 6.2 for C_{t+1} and plug in the result into equation 6.4 to get

$$\frac{1-\gamma C_t}{\gamma D_t} = p_t - \frac{(1-\delta)p_{t+1}}{1+r_{t+1}} \tag{6.4}$$

d) *Interpret the first order conditions.*

The MRS between durable and non-durable consumption is equal to their price ratio/

e) *Suppose now that the purchase price of durables and the interest rates are constant so that, $p_s = p$ and $r_s = r$. In addition, assume $\beta = \frac{1}{1+r}$ and $Y_t = 0.9Y_{t-1} + \epsilon_t$ where $E[\epsilon_t] = 0$, $E[\epsilon_t\epsilon_s] = 0$ for $s \neq t$, and $E[\epsilon_t^2] = \sigma^2$. Explain what happens to the path of the flow of services from durables over time. Briefly relate your findings to Hall's random walk model for consumption.*

We can simply equation 6.4 from the assumptions in the question allowing us to simplifying the FOC's (equations 6.2 and 6.4 respectively) to be

$$C_t = C_{t+1} \tag{6.5}$$
$$\frac{1-\gamma C_t}{\gamma D_t} = p\frac{(r+\delta)}{1+r} \tag{6.6}$$

For simplicity I will denote the constant rental price as $\alpha = p\frac{(r+\delta)}{1+r}$ so $\alpha D_t = \frac{1-\gamma}{\gamma}C_t$. Then we need to write down the inter-temporal budget constraint.

$$\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(C_s + \alpha D_s) = (1+r)B_t + (1-\delta)pD_{t-1} + \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(Y_s)$$

Because we know the consumption path of non-durables from equation 6.5 , we also can solve for the path of durables and simplify to

$$\frac{1+r}{r}\left(\frac{\alpha}{1-\gamma}D_t\right) = (1+r)B_t + (1-\delta)pD_{t-1} + \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(Y_t)$$

Then using the AR process given in the question for Y_t we can simplify more to

$$\frac{1+r}{r}\left(\frac{\alpha}{1-\gamma}D_t\right) = (1+r)B_t + (1-\delta)pD_{t-1} + \frac{1+r}{r+0.1}.9Y_{t-1}$$

Solve for D_t to get

$$D_t = \frac{1-\gamma}{\alpha}\left(rB_t + r(1-\delta)pD_{t-1} + \frac{r}{r+0.1}.9Y_{t-1}\right)$$

which shows that D_t follows a random walk process proportional to the stochastic process of Y_t if we assume there is no initial asset or durable stock.

f) *Explain what happens to the paths of the ow of services from durables and the purchases of durables if there is a positive shock to endowment at time t. Justify your answer.*

Consumption of durables would increase in the short run but would move back to normal over time. This is because the AR coefficient is less than one.

g) *How would your answers to 4f change if the process for endowment was given by* $Y_t - Y_{t-1} = 0.9(Y_{t-1} - Y_{t-2}) + \epsilon_t$.

We would plug in $Y_t = 1.9Y_{t-1} - 0.9(Y_{t-2}) + \epsilon_t$ instead. I think this means the shock is permanent. The AR coefficient on the first lag is now greater than one, so the shock would permanently increase the stock. Email me if you think I'm missing something on this part.

7 Real Business Cycle June 2017 #4

Consider a Real Business Cycle model that can be characterized as the solution to the planner's problem:

$$\max E \sum_{t=0}^{\infty} \beta^t \left(lnC_t - \varphi N_t \right)$$

where parameter $\varphi > 0$captures the dis-utility from working, ln takes the natural log of consumption, β is the time preference parameter, and E is the expectations operator. The control variables are a sequence of consumption and labor choices along with the future capital stock. The initial stock of capital, K, at time t=0 is given. For future periods, capital evolves according to the following law of motion:

$$K_{t+1} = A_t K_t^{\alpha} N_t^{1-\alpha} - C_t + (1-\delta) K_t$$

where the production function is Cobb-Douglas and capital depreciates at rate δ. Productivity shocks drive the cycle: A_t evolves according to the following equation:

$$\ln A_t = \rho \ln A_{t-1} + e_t$$

with ρ between zero and one and e_t distributed as iid white noise. There is no population or technology growth.

a) *Write down the Lagrangian and find the first order conditions (Euler equations) that characterize the solution to the planners problem.*

One way to do the Lagrangian is

$$\mathcal{L} = \max_{C_t, N_t} E_0 \Big[\sum_{t=0}^{\infty} \beta^t (\ln C_t - \phi N_t) - \lambda_t [K_{t+1} - A_t K_t^{\alpha} N_t^{1-\alpha} - C_t + (1-\delta) K_t \Big]$$

Then we take first order conditions at time t and $t+1$ for all choice variables (Except capital because we have the law of motion already)

$$\frac{\delta L}{\delta C_t} : \frac{\beta^t}{C_t} = -\lambda_t$$

$$\frac{\delta L}{\delta C_{t+1}} : \frac{\beta^{t+1}}{C_{t+1}} = -\lambda_{t+1}$$

$$\frac{\delta L}{\delta N_t} : -\phi \beta^t = (1-\alpha) A_t K_t^{\alpha} N_t^{-\alpha} \lambda_t$$

$$\frac{\delta L}{\delta N_{t+1}} : -\phi \beta^{t+1} = (1-\alpha) A_{t+1} K_{t+1}^{\alpha} N_{t+1}^{-\alpha} \lambda_{t+1}$$

$$\frac{\delta L}{\delta K_{t+1}} : \lambda_t = \lambda_{t+1} \left(\alpha A_t K_t^{\alpha-1} N_t^{1-\alpha} - (1-\delta) \right)$$

Note these are all technically stochastic values based on the technological law of motion. This gives us

$$\frac{\lambda_t}{\lambda_{t+1}} = \left(\alpha A_t K_t^{\alpha-1} N_t^{1-\alpha} - (1-\delta) \right)$$

which can simplify to

$$\frac{C_{t+1}}{C_t} = \beta \left(\alpha A_t K_t^{\alpha-1} N_t^{1-\alpha} - (1-\delta) \right)$$

and

$$C_t = -\frac{(1-\alpha)}{\phi} \Big(\frac{K_t}{N_t} \Big)^{\alpha}$$

This could probably be simplified more.

8 A Simple Government

Suppose the government proposes to undertake a large infrastructure project that does not provide any direct benefit to the economy (e.g. digging ditches or building bridges to nowhere). Due to previous surpluses, the government does not have to tax to pay for the project. However, the government does need to hire workers to build the infrastructure.

Assume that a profit maximizing representative firm and the government have access to the same production technology, which makes use of only units of labor (L):

$$Y_{G,F} = L_{G,F}^{\alpha}$$

where $0 < \alpha < 1$; G and F denote production and labor employed by the government and firm, respectively, and both take the per unit real wage (W) to hire labor as given. The representative consumer also takes wages as given and chooses consumption (C) and labor supply (LS) in order to maximize utility:

$$U(C, L) = \ln C + \ln(1 - L)$$

subject to a budget constraint

$$C = WL$$

The goods market need not clear, and the firm simply retains any profits.

a) *What is the short-run effect of increasing government output on hours worked?*

First we need to find equilibrium labor supply. For the household this is fixed at 1/2. I plug in the budget constraint and solve.

$$\max_{L} \ln WL + \ln(1 - L)$$

$$\frac{\partial U}{\partial L} = \frac{W}{WL} + \frac{-1}{1 - L} = 0$$

$$L_s = 1/2$$

Therefore labor is fixed, and government output has no effect on hours worked.

b) *What is the short-run effect of increasing government output on equilibrium wages?* Now for the demand side we need the firm and government labor demand. Firm's maximize profit so

$$\max_{L} L_F^{\alpha} - WL$$

$$\frac{\partial \Pi}{\partial L} = \alpha L_F^{\alpha - 1} - W = 0$$

$$L_F = \frac{\alpha}{W}^{1-\alpha}$$

For the government, they just select their output level, so

$$L_G = Y_G^{\frac{1}{\alpha}}$$

Now find equilibrium wage by setting labor supply equal to labor demand

$$L_D = L_G + L_F = LS$$

$$Y_G^{\frac{1}{\alpha}} + \frac{\alpha}{W}^{1-\alpha} = 1/2$$

Then solve for W.

$$W^* = \alpha/(\frac{1}{2} - Y_G^{\frac{1}{\alpha}})^{1-\alpha}$$

Here we can see that $\frac{\partial W}{\partial Y_G} > 0$

c) *What is the short-run effect of increasing government output on consumption*

We know labor is fixed at $1/2$, so $C = 0.5W$, which mean $\frac{\partial C}{\partial Y_G} > 0$

d) *What is the short-run effect of increasing government output on production?*

Government output crowds out firm output in the short run because labor supplied is fixed.

9 Problems to work on

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